The alternating group: branching rules, products and plethysms for ordinary and spin representations

1981 J. Phys. A: Math. Gen. 14 1835

(http://iopscience.iop.org/0305-4470/14/8/011)

J. Phys. A: Math. Gen. 14 (1981) 1835–1848. Printed in Great Britain

The alternating group: branching rules, products and plethysms for ordinary and spin representations

Luan Dehuai† and B G Wybourne

Department of Physics, University of Canterbury, Christchurch, New Zealand

Received 23 October 1980, in final form 5 January 1981

Abstract. Practical algorithms are given for calculating all possible Kronecker products involving the spin and ordinary representations of the alternating group A_n and for the resolution of Kronecker squares into their symmetric and antisymmetric parts. The representations of A_n are classified as to their orthogonal, symplectic or complex characters. Branching rules for $S_n \downarrow A_n$ and $A_n \downarrow A_{n-1}$ are given. Throughout, the emphasis is on obtaining results that obviate the need for explicit character tables and on presenting the results in an n-independent manner.

1. Introduction

We have recently indicated how the branching rules, Kronecker products and plethysms involving the spin representations of the symmetric group S_n may be developed in an essentially n-independent manner and without the explicit use of character tables (Luan and Wybourne 1981, referred to as I). In this paper we extend the results of I to the ordinary and spin representations of the alternating group A_n, the group of even permutations. The group A_n is of order $n!/2$ and is a subgroup of index 2 of S_n.

From a mathematical viewpoint, the A_n groups assume a special significance as a consequence of the result that for $n \neq 4$ the groups A_n are necessarily simple (Ledermann 1976). Thus the groups A_n form a great class of finite simple groups. In physics the isomorphisms $C_3 \sim A_3$, $T \sim A_4$ and $I \sim A_5$ are well known in solid state and molecular physics (cf Lax 1974).

The ordinary irreducible representations (irreps) of A_n were studied long ago (Frobenius 1901), and more recently from the point of view of induced representations (Puttaswamaiah and Robinson 1964). A partial study of the projective irreps of A_n was made by Schur (1911). Apart from a few particular cases, very little attention seems to have been devoted to the study of the spin representations of A_n.

In this paper we shall first review some of the relevant aspects of the symmetric and alternating groups. The reduced notation developed in I for S_n is extended to A_n, leading to an essentially n-independent treatment of the properties of the irreps of A_n. Branching rules for $S_n \downarrow A_n$ are developed. The difference characters for the irreps of A_n are established, and used to establish a series of algorithms for evaluating Kronecker

† On leave from Department of Applied Mathematics, Beijing Polytechnic University, Beijing, China.

products and plethysms of the spin and ordinary irreps of A_n. In the concluding section the systematic classification of the irreps of A_n is given. Throughout we follow closely the notations established in I.

2. The groups S_n and A_n

The cycle structure of a given class of conjugate permutations in S_n may be designated as

$$(1^{\nu_1}2^{\nu_2}\ldots n^{\nu_n}) \tag{1}$$

where ν_1 is the number of 1-cycles etc and

$$1\nu_1+2\nu_2+\ldots+n\nu_n=n. \tag{2}$$

Cycles structures involving an even number of even length cycles correspond to even permutations, while all other permutations are odd. To each solution of (2) for positive integers there corresponds a class in S_n (cf Hamermesh 1962).

For later convenience we shall adopt the convention of listing the cycle structures in order of their decreasing length and omit all cycles with exponents $\nu_i=0$. Parentheses will be used to enclose the cycle sequences. Thus in S_4 we designate the classes as (1^4), (2^2), (31), (21^2), (4) with the first three classes involving even permutations only.

The classes of S_n are all ambivalent, i.e. every class contains the inverses of its elements (Sharp *et al* 1975). As a consequence all the ordinary irreps of S_n are real and S_n is said to be an ambivalent group.

The classes of A_n involve only even permutations. All classes of S_n involving only even permutations remain as classes of A_n, with the important exception of those classes for which the even permutations involve only odd cycles of unequal length. In those cases the class splits into two classes of conjugate elements of A_n, each with half the number of elements (Frobenius 1901, Boerner 1970). The splitting classes of A_n will be designated as $(p_1p_2\ldots p_k)_+$ and $(p_1p_2\ldots p_k)_-$ where the p_i and are all odd and

$$p_1>p_2>\ldots>p_k>0 \tag{3}$$

with

$$p_1+p_2+\ldots+p_k=n. \tag{4}$$

Thus in A_4 we have the classes (1^4), (2^2), $(31)_+$, $(31)_-$.

It is well known from number theory (cf Hardy and Wright 1954) that the number of partitions of n into odd and unequal parts is equal to the number of its self-associated partitions. Thus with every self-associated partition $(\lambda_1\lambda_2\ldots)$ we have the partition $(p_1p_2\ldots)$ where

$$p_i=2\lambda_i-2i+1 \tag{5}$$

and we have the constraints of (3) and (4). The number of self-associated partitions for given n is readily seen to be

$$\sum_m p_m[(n-m^2)/2] \tag{6}$$

where $p_m[k]$ is the number of partitions of k into at most m parts and the summation is over $m=2,4,\ldots$ for n even and $m=1,3,\ldots$ for n odd.

The splitting classes of A_n are ambivalent for $n = 2, 5, 6, 10$ and 14—all other A_n are non-ambivalent (Sharp *et al* 1975).

The notion of an irreducible projective representation (IPR) plays an important role in the theory of the spin representations of S_n and A_n. We now consider a few of the relevant properties of IPRs (cf Schur 1911, Curtis and Reiner 1962, Dornhoff 1971).

Let G be a finite group, k a field and V a finite-dimensional k-vector space. A projective representation of G on V is a mapping $T: \mathrm{G} \to \mathrm{GL}(V)$ such that for all $x, y \in G$

$$T(x)T(y) = \alpha(x, y)T(xy), \qquad \alpha(x, y) \in k. \tag{7}$$

The function $\alpha: \mathrm{G} \times \mathrm{G} \to k$ is called a factor set of T and T an IPR if V has no proper subspace invariant under all $T(x)$, $x \in \mathrm{G}$. Furthermore

$$\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z) \qquad \text{for all } x, y, z \in \mathrm{G}. \tag{8}$$

Two factor sets α and β of G are termed equivalent if there is a function $\gamma: \mathrm{G} \to k$ such that

$$\alpha(x, y) = \beta(x, y)\gamma(x)\gamma(y)\gamma(xy)^{-1} \qquad \text{for all } x, y, z \in \mathrm{G}. \tag{9}$$

The set $H^2(\mathrm{G}, k)$ of all equivalence classes under (9), with multiplication $\{\alpha\}\{\beta\} = \{\alpha\beta\}$ well defined, forms an Abelian group of equivalence classes of the factor sets and is known as the Schur multiplier of G over k. For S_n we have (Davies and Morris 1974)

$$H^2(\mathrm{S}_n, \mathbb{C}^*) = C_2 = \{(r)\} \qquad (n \geqslant 4) \tag{10}$$

where $\mathbb{C}^*$ denotes the non-zero complex numbers and if $r = 1$, the irrep T of S_n will be called an ordinary irrep while if $r = -1$, T will be called an IRP or spin irrep of S_n.

The centraliser $C(x)$ of an element $x \in \mathrm{G}$ is the collection of elements $s \in \mathrm{G}$ such that $sxs^{-1} = x$. If α is a factor set of G, an element $x \in \mathrm{G}$ will be termed an α-regular element if

$$\alpha(x, s) = \alpha(s, x) \tag{11}$$

for all s in the centraliser of x in G. If x is α-regular then every element which is conjugate to x in G is α-regular, and hence we may speak of an α-regular class.

The number of distinct inequivalent IPRs of G with the factor set α is equal to the number of α-regular classes of G, and

$$\sum_i n_i^2 = g \tag{12}$$

where n_i are the dimensions of the inequivalent IPRs and g is the order of G.

For S_n the α-regular classes fall into two categories: (1) even permutation classes containing only cycles of odd order; (2) odd permutation classes containing cycles of unequal orders. Thus for S_6 we have the six α-regular classes

$$(1^6), (51), (3^2), (31^3) \qquad \text{even,}$$

$$(6), (321) \qquad \text{odd.}$$

In the case of A_n the even α-regular classes of S_n remain as α-regular classes of A_n, though among them there may be splitting classes. In addition, there are the even classes involving cycles of unequal orders. These latter classes are α-irregular in S_n but α-regular in A_n. Thus for A_6 we have six α-regular classes

$$(1^6), (51)_+, (51)_-, (3^2), (31^3), (42)$$

where the α-regular (51) class of S_6 has split and the α-irregular (42) class of S_6 is α-regular in A_6.

The ordinary irreps of S_n are labelled by ordered partitions $[\lambda]$ of n while the spin irreps are labelled by ordered partitions $[\lambda]'$ of n into unequal parts. The ordinary irreps of S_n are said to be self-associated if $[\lambda] \equiv [\tilde{\lambda}]$ (where $(\tilde{\lambda})$ is the partition conjugate to (λ)) and will be designated as $[\lambda]^{\dagger}$. For all other ordinary irreps the pairs $[\lambda]$ and $[\tilde{\lambda}]$ are associated and (λ) will be taken to be the partition of greatest weight of (λ) and $(\tilde{\lambda})$. The spin irreps are self-associated if $n-k$ is even (k being the number of parts of $[\lambda]'$) and will be designated as $[\lambda]'^{\dagger}$—all other spin irreps of S_n form associated pairs $[\lambda]'$ and $[\tilde{\lambda}]'$.

Under $S_n \downarrow A_n$ the pair of associated irreps of S_n become equivalent irreps of A_n, while self-associated irreps of S_n split into two conjugate irreps of A_n of the same degree (Frobenius 1901, Read 1977). As a consequence, we shall label the irreps of A_n by partitions of n. For the ordinary irreps, if $[\lambda] \neq [\tilde{\lambda}]$ then we use only the partition of greatest weight, while if $[\lambda] \equiv [\tilde{\lambda}]$ we have two conjugate irreps designated as $[\lambda]_+$ and $[\lambda]_-$. The spin irreps of A_n are labelled by partitions $[\tilde{\lambda}]'$ of n into unequal parts. If $n-m$ is even there are two conjugate irreps $[\lambda]'_+$ and $[\lambda]'_-$.

Following I, we shall frequently use $[\lambda]^{\dagger} = [\lambda] + [\tilde{\lambda}]$ etc for designating pairs of associated irreps of S_n.

3. $S_n \downarrow A_n$ branching rules and reduced notation

In terms of the notation just outlined we may write the $S_n \downarrow A_n$ branching rules as (Frobenius 1901, Schur 1911)

$$S_n \downarrow A_n,$$

$$[\lambda]^{\dagger} \downarrow [\lambda]_+ + [\lambda]_- \qquad \text{when } [\lambda] \equiv [\tilde{\lambda}], \tag{13a}$$

$$[\lambda]^{\dagger} \downarrow 2[\lambda] \qquad \text{when } [\lambda] \neq [\tilde{\lambda}], \tag{13b}$$

$$[\lambda]'^{\dagger} \downarrow [\lambda]'_+ + [\lambda]'_- \qquad \text{when } n-k \text{ even} \tag{14a}$$

$$[\lambda]'^{\dagger} \downarrow 2[\lambda]' \qquad \text{when } n-k \text{ odd.} \tag{14b}$$

A reduced notation for labelling the irreps of S_n in an n-independent manner was developed in I. The ordinary irreps of S_n usually labelled by the n-dependent symbol

$$[\lambda] \equiv [n-m, \mu_1, \mu_2, \ldots, \mu_r],$$

with (μ) being a partition of m were labelled by the n-independent symbol $\langle\mu\rangle \equiv \langle\mu_1\mu_2\ldots\mu_r\rangle$. The spin irreps of S_n were labelled in a similar manner, with a prime being added to distinguish them from ordinary irreps of S_n. This reduced notation may be carried over to A_n to give the $S_n \downarrow A_n$ branching rules in an essentially n-independent form as

$$S_n \downarrow A_n,$$

$$\langle\mu\rangle^{\dagger} \downarrow \langle\mu\rangle_+ + \langle\mu\rangle_- \qquad \text{when } \langle\mu\rangle \equiv \langle\tilde{\mu}\rangle, \tag{15a}$$

$$\langle\mu\rangle^{\dagger} \downarrow 2\langle\mu\rangle \qquad \text{when } \langle\mu\rangle \neq \langle\tilde{\mu}\rangle, \tag{15b}$$

$$\langle\mu\rangle'^{\dagger} \downarrow \langle\mu\rangle'_+ + \langle\mu\rangle'_- \qquad \text{when } n-r \text{ odd} \tag{16a}$$

$$\langle\mu\rangle'^{\dagger} \downarrow 2\langle\mu\rangle' \qquad \text{when } n-r \text{ even.} \tag{16b}$$

In using the above results it is essential to remember that the self-association of irreps is an n-dependent property.

4. The $\mathrm{A}_n \downarrow \mathrm{A}_{n-1}$ branching rules

The branching rules for $\mathrm{S}_n \downarrow \mathrm{S}_{n-1}$ were developed in I. Knowing these, together with the $\mathrm{S}_n \downarrow \mathrm{A}_n$ rules just outlined, leads immediately to the branching rules for $\mathrm{A}_n \downarrow \mathrm{A}_{n-1}$. For the ordinary irreps of A_n we have in the reduced notation

$$\langle\mu\rangle \downarrow \langle\mu\rangle + \langle\mu/1\rangle, \qquad \langle\mu\rangle \neq \langle\tilde{\mu}\rangle, \tag{17a}$$

$$\langle\mu\rangle_\pm \downarrow \tfrac{1}{2}(\langle\mu\rangle + \langle\mu/1\rangle), \qquad \langle\mu\rangle \equiv \langle\tilde{\mu}\rangle. \tag{17b}$$

The terms on the right, $\langle\rho\rangle$, are to be taken as $\langle\rho\rangle_+ + \langle\rho\rangle_-$ if $\langle\rho\rangle \equiv \langle\tilde{\rho}\rangle$.
Thus we have

$$\langle 1^2\rangle \downarrow \langle 1^2\rangle + \langle 1\rangle,$$

leading in $n = 5$ to

$$[31^2]_\pm \downarrow \tfrac{1}{2}([21^2] + [31]) = [31]$$

and in $n = 6$ to

$$[41^2] \downarrow [31^2] + [41] = [31^2]_+ + [31^2]_- + [41].$$

Likewise we find for the spin irreps of A_n under $\mathrm{A}_n \downarrow \mathrm{A}_{n-1}$ for $n - r$ *even*

$$2\langle\mu\rangle' \downarrow (\langle\mu\rangle'^{\dagger} + 2\langle\mu/1\rangle'^{\dagger} - \delta_{\lambda_r,1}\langle\mu_1, \ldots, \mu_{r-1}\rangle'^{\dagger}) \tag{18a}$$

while for $n - r$ *odd*

$$\langle\mu\rangle'_\pm \downarrow \tfrac{1}{2}(\langle\mu\rangle'^{\dagger} + \langle\mu/1\rangle'^{\dagger}\delta_{\lambda_r,1}\langle\mu_1, \ldots, \mu_{r-1}\rangle'_\pm - \delta_{\lambda_r,1}\langle\mu_1, \ldots, \mu_{r-1}\rangle'_\mp) \tag{18b}$$

where we use the † to indicate self-associated spin irreps of S_n and restrict to A_n using (16a) and (16b), i.e.

$$\langle\mu\rangle'^{\dagger} \equiv \langle\mu\rangle' + \langle\tilde{\mu}\rangle' \qquad \text{or} \qquad \langle\mu\rangle' = \langle\tilde{\mu}\rangle. \tag{19}$$

Thus under $\mathrm{A}_n \downarrow \mathrm{A}_{n-1}$ we have for $n - r$ even

$$2\langle 421\rangle' \downarrow 2\langle 421\rangle'^{\dagger} + 2\langle 321\rangle'^{\dagger} + \langle 42\rangle'^{\dagger}$$

and hence for $\mathrm{A}_{13} \downarrow \mathrm{A}_{12}$

$$[6421]' \downarrow [5421]'_+ + [5421]'_- + [6321]'_+ + [6321]'_- + [642]',$$

while for $n - r$ odd

$$\langle 421\rangle_\pm \downarrow \tfrac{1}{2}(\langle 421\rangle'^{\dagger} + \langle 321\rangle'^{\dagger} + \langle 42\rangle^{\dagger\prime} + \langle 42\rangle'_\pm - \langle 42\rangle'_\mp)$$

and hence for $\mathrm{A}_{12} \downarrow \mathrm{A}_{11}$

$$[5421]'_\pm \downarrow \tfrac{1}{2}(2[5321]' + 2[542]'_\pm) = [5321]' + [542]'_\pm.$$

5. Difference characters for A_n

The simple characters of S_n which are not self-associated are also simple characters of A_n. Each self-associated character of S_n is the sum of two simple characters of A_n. For the ordinary irreps, the characters of A_n are found by taking half the value of the characteristics for S_n, except for the splitting classes $(\rho)_\pm \equiv (p_1 p_2 \ldots p_k)_\pm$ where $p_i = 2\lambda_i - 2i + 1$ (cf equations (3)–(5)). In this case the characteristics of the splitting classes in A_n are given by (Frobenius 1901)

$$\chi^{[\lambda]_+}_{(p)_\pm} = \tfrac{1}{2}[(-1)^{(n-k)/2} \pm \mathrm{i}^{(n-k)/2}(p_1 p_2 \ldots p_k)^{1/2}] \tag{20a}$$

and

$$\chi^{[\lambda]_-}_{(p)_\pm} = \tfrac{1}{2}[(-1)^{(n-k)/2} \mp \mathrm{i}^{(n-k)/2}(p_1 p_2 \ldots p_k)^{1/2}]. \tag{20b}$$

If the difference character is defined as (cf Read 1977)

$$\chi^{[\lambda]''}_{(\rho)} = \chi^{[\lambda]_+}_{(\rho)} - \chi^{[\lambda]_-}_{(\rho)} \tag{21}$$

then it follows from (17a) and (17b) that

$$\chi^{[\lambda]''}_{(p)_\pm} = \pm \mathrm{i}^{(n-k)/2}(p_1 p_2 \ldots p_k)^{1/2} \tag{22}$$

if $(\rho) \equiv (p)_\pm$ and vanishes for all other classes.

In the case of the spin irreps of S_n, those irreps labelled by partitions (λ) into k unequal odd parts, with $n-k$ even, are self-associated and split into two conjugate irreps $[\lambda]'_+$ and $[\lambda]'_-$ of A_n. Here we find for the splitting classes $(\lambda)_\pm$

$$\chi^{[\lambda]'_+}_{(\lambda)_\pm} = \tfrac{1}{2}[(-1)^{(k-n(\mathrm{mod}\ 2))/2} \pm \mathrm{i}^{(n-k)/2}(\lambda_1 \lambda_2 \ldots \lambda_k)^{1/2}] \tag{23a}$$

and

$$\chi^{[\lambda]'_-}_{(\lambda)_\pm} = \tfrac{1}{2}[(-1)^{(k-n(\mathrm{mod}\ 2))/2} \mp \mathrm{i}^{(n-k)/2}(\lambda_1 \lambda_2 \ldots \lambda_k)^{1/2}] \tag{23b}$$

and hence

$$\chi^{[\lambda]'''}_{(\lambda)_\pm} = \pm \mathrm{i}^{(n-k)/2}(\lambda_1 \lambda_2 \ldots \lambda_k)^{1/2}, \tag{24}$$

while for the class (λ) where all parts of (λ) are unequal and one or more even

$$\chi^{[\lambda]'_\pm}_{(\lambda)} = \pm \mathrm{i}^{(n-m)/2}(\lambda_1 \lambda_2 \ldots \lambda_k)^{1/2}/2 \tag{25}$$

and hence

$$\chi^{[\lambda]'''}_{(\lambda)} = \pm \mathrm{i}^{(n-m)/2}(\lambda_1 \lambda_2 \ldots \lambda_k)^{1/2}. \tag{26}$$

In all other classes the spin characteristics for the conjugate irreps $[\lambda]'_+$ and $[\lambda]'_-$ of A_n are simply half of their corresponding values in S_n, and the difference characteristics vanish.

It follows from the above that the associated characters of S_n decompose into real characters of A_n, while the self-associated characters of S_n decompose into a pair of real conjugate characters of A_n if $n-k = 0(\mathrm{mod}\ 4)$, otherwise we obtain a pair of complex characters of A_n.

By way of an example of the notation developed here we have given the spin characteristics for the α-regular classes of A_8 in table 1.

Table 1. Spin characteristics for the α-regular classes of A_8.

Class	(1^8)	(31^5)	(51^3)	(3^21^2)	$(71)_+$	$(71)_-$	(62)	$(53)_+$	$(53)_-$
Order	1	112	1344	1120	2880	2880	3360	1344	1344
$[8]'$	8	4	2	2	1	1	0	1	1
$[71]'_+$	24	6	1	0	$\frac{1}{2}(-1-i\sqrt{7})$	$\frac{1}{2}(-1+i\sqrt{7})$	0	-1	-1
$[71]'_-$	24	6	1	0	$\frac{1}{2}(-1+i\sqrt{7})$	$\frac{1}{2}(-1-i\sqrt{7})$	0	-1	-1
$[62]'_+$	56	4	-1	-1	0	0	$-i\sqrt{3}$	1	1
$[62]'_-$	56	4	-1	-1	0	0	$+i\sqrt{3}$	1	1
$[53]'_+$	56	-2	-1	2	0	0	0	$\frac{1}{2}(-1+i\sqrt{15})$	$\frac{1}{2}(-1-i\sqrt{15})$
$[53]'_-$	56	-2	-1	2	0	0	0	$\frac{1}{2}(-1-i\sqrt{15})$	$\frac{1}{2}(-1+i\sqrt{15})$
$[521]'$	64	-4	1	-2	1	1	0	-1	-1
$[431]'$	48	-6	2	0	-1	-1	0	1	1

6. Kronecker products in A_n

The analysis of the Kronecker products of irreps of A_n given here draws heavily upon the methods developed in I, but with a rather more extensive use of the properties of difference characters (Littlewood 1950, Wybourne 1970). Our results are summarised in seven algorithms which cover all possible cases. Most of the evaluations are made by first resolving a Kronecker product in S_n and then using the $S \downarrow A_n$ branching rules together with the properties of the difference characters to assign the A_n irreps to the appropriate A_n product.

The first algorithm covers all cases that do not involve members of a conjugate pair of A_n irreps.

Algorithm 1.

(1) To resolve $\langle\mu\rangle\langle\nu\rangle$, $\langle\mu\rangle\langle\nu\rangle'$ or $\langle\mu\rangle'\langle\nu\rangle'$ make the replacements

$$\langle\mu\rangle\langle\nu\rangle \rightarrow \langle\mu\rangle\langle\nu\rangle \tag{27a}$$

$$\langle\mu\rangle\langle\nu\rangle' \rightarrow \langle\mu\rangle\langle\nu\rangle'^{\dagger}/2, \tag{27b}$$

$$\langle\mu\rangle'\langle\nu\rangle' \rightarrow \langle\mu\rangle'^{\dagger}\langle\nu\rangle'^{\dagger}/4. \tag{27c}$$

(2) In each case the right-hand side involves S_n Kronecker products—resolve these using the algorithms given in I.

(3) Restrict the resulting S_n irreps to those of A_n using (15*a*)–(16*b*).

The second algorithm resolves the products $\langle\mu\rangle\langle\nu\rangle_\pm$ for ordinary irreps of A_n.

Algorithm 2.

(1) Evaluate $\langle\mu\rangle\langle\nu\rangle$ for S_n giving

$$\langle\mu\rangle\langle\nu\rangle = g^{\rho}_{\mu\nu}\langle\rho\rangle$$

and restrict the right-hand side to A_n using (15*a*) and (15*b*).

(2) Divide the coefficients associated with every term found in (1) by two. The integral part of the resulting coefficients is the number of times its corresponding irrep occurs in $\langle\mu\rangle\langle\nu\rangle_+$ and in $\langle\mu\rangle\langle\nu\rangle_-$. If there is no residue the resolution is complete.

(3) The only possible residue will be a term

$$\langle\nu\rangle^{\dagger}=\langle\nu\rangle_{+}+\langle\nu\rangle_{-}.$$

If the characteristic $\chi^{\langle\mu\rangle}_{\langle p\rangle}=+1$ then $\langle\nu\rangle_+$ is assigned to $\langle\mu\rangle\langle\nu\rangle_+$ and $\langle\nu\rangle_-$ to $\langle\mu\rangle\langle\nu\rangle_-$, while if $\chi^{\langle\mu\rangle}_{\langle p\rangle}=-1$ the opposite assignment is made.

The third algorithm treats the case $\langle\mu\rangle_{\pm}\langle\nu\rangle'$.

Algorithm 3.

(1) Evaluate $\langle\mu\rangle\langle\nu\rangle'^{\dagger}$ for S_n using algorithm 2 of I to give

$$\langle\mu\rangle\langle\nu\rangle'^{\dagger}=g^{\rho}_{\mu\nu}\langle\rho\rangle'^{\dagger}$$

and restrict the above results to A_n using (16*a*) and (16*b*) to give $(\langle\mu\rangle_{+}+\langle\mu\rangle_{-})\langle\nu\rangle'=\frac{1}{2}g^{\rho}_{\mu\nu}\langle\rho\rangle'^{\dagger}$.

(2) Divide the coefficients associated with every term found in (1) by two. The integral part of the resulting coefficients is the number of times its corresponding irrep occurs in $\langle\mu\rangle_+\langle\nu\rangle'$ and $\langle\mu\rangle_-\langle\nu\rangle'$. If there is no residue the resolution is complete.

(3) The only possible residue will be a term

$$\langle p\rangle'^{\dagger}=\langle p\rangle'_{+}+\langle p\rangle'_{-}.$$

If the characteristic $\chi^{\langle\nu\rangle'}_{\langle p\rangle}=+1$ then $\langle p\rangle'_+$ is assigned to $\langle\mu\rangle_+\langle\nu\rangle'$ and $\langle p\rangle'_-$ to $\langle\mu\rangle_-\langle\nu\rangle'$, while if $\chi^{\langle\nu\rangle'}_{\langle p\rangle}=-1$ the opposite assignment is made.

The fourth algorithm covers the case $\langle\mu\rangle\langle\nu\rangle'_{\pm}$.

Algorithm 4.

(1) Evaluate $\langle\mu\rangle\langle\nu\rangle'^{\dagger}$ for S_n using algorithm 2 of I to give

$$\langle\mu\rangle\langle\nu\rangle'^{\dagger}=g^{\rho}_{\mu\nu}\langle\rho\rangle'^{\dagger}$$

and restrict the right-hand side to A_n using (16*a*) and (16*b*).

(2) Divide the coefficients associated with every term found in (1) by two. The integral part of the resulting coefficients is the number of times its corresponding irrep occurs in $\langle\mu\rangle\langle\nu\rangle'_+$ and in $\langle\mu\rangle\langle\nu\rangle'_-$. If there is no residue the resolution is complete.

(3) The only possible residue will be a term

$$\langle\nu\rangle'^{\dagger}=\langle\nu\rangle'_{+}+\langle\nu\rangle'_{-}.$$

If the characteristic $\chi^{\langle\mu\rangle}_{\langle\nu\rangle}=1$ then $\langle\nu\rangle'_+$ is assigned to $\langle\mu\rangle\langle\nu\rangle'_+$ and $\langle\nu\rangle'_-$ to $\langle\mu\rangle\langle\nu\rangle'_-$, while if $\chi^{\langle\mu\rangle}_{\langle\nu\rangle}=-1$ the opposite assignment is made.

The next three algorithms require the use of difference characters. Let

$$\langle\mu\rangle^{\dagger}=\langle\mu\rangle_{+}+\langle\mu\rangle_{-} \tag{28a}$$

and

$$\langle\mu\rangle''=\langle\mu\rangle_{+}-\langle\mu\rangle_{-}; \tag{28b}$$

then generally

$$\langle\mu\rangle_{\pm}\langle\nu\rangle_{\pm}=\tfrac{1}{4}[\langle\mu\rangle^{\dagger}\langle\nu\rangle^{\dagger}\pm\langle\mu\rangle^{\dagger}\langle\nu\rangle''\pm\langle\mu\rangle''\langle\nu\rangle^{\dagger}+\langle\mu\rangle''\langle\nu\rangle''] \tag{29a}$$

and

$$\langle\mu\rangle_{\pm}\langle\nu\rangle_{\mp}=\tfrac{1}{4}[\langle\mu\rangle^{\dagger}\langle\nu\rangle^{\dagger}\pm\langle\mu\rangle^{\dagger}\langle\nu\rangle''\mp\langle\mu\rangle''\langle\nu\rangle^{\dagger}-\langle\mu\rangle''\langle\nu\rangle''].\tag{29b}$$

For ordinary irreps of A_n

$$\langle\mu\rangle''\langle\nu\rangle=\langle\mu\rangle\langle\nu\rangle''=\langle\mu\rangle''\langle\nu\rangle''=0 \qquad \text{if } \mu\neq\nu\tag{30a}$$

and hence

$$\langle\mu\rangle_{\pm}\langle\nu\rangle_{\pm}=\langle\mu\rangle_{\pm}\langle\nu\rangle_{\mp}=\tfrac{1}{4}\langle\mu\rangle\langle\nu\rangle\tag{30b}$$

while if $\mu\equiv\nu$ then

$$\langle\mu\rangle_{\pm}^{2}=\tfrac{1}{4}[\langle\mu\rangle^{\dagger 2}\pm 2\langle\mu\rangle^{\dagger}\langle\mu\rangle''+\langle\mu\rangle''^{2}]\tag{31a}$$

and

$$\langle\mu\rangle_{+}\langle\mu\rangle_{-}=\tfrac{1}{4}[\langle\mu\rangle^{\dagger 2}-\langle\mu\rangle''^{2}].\tag{31b}$$

For the spin irreps of A_n there are two distinct cases; (a) spin irreps labelled by partitions into unequal parts with one or more even; (b) spin irreps labelled by partitions into unequal odd parts.

In case (a) we find

$$\langle\nu\rangle'^{\dagger}\langle\nu\rangle'''=0$$

and hence

$$\langle\nu\rangle'^{2}_{\pm}=\tfrac{1}{4}[\langle\nu\rangle'^{\dagger 2}+\langle\nu\rangle'''^{2}]\tag{32a}$$

and

$$\langle\nu\rangle'_{+}\langle\nu\rangle'_{-}=\tfrac{1}{4}[\langle\nu\rangle'^{\dagger 2}-\langle\nu\rangle'''^{2}],\tag{32b}$$

while in case (b) we find

$$\langle\nu\rangle'^{\dagger}\langle\nu\rangle'''\neq 0$$

and hence

$$\langle\nu\rangle'^{2}_{\pm}=\tfrac{1}{4}[\langle\nu\rangle'^{\dagger 2}\pm 2\langle\nu\rangle'^{\dagger}\langle\nu\rangle'''+\langle\nu\rangle'''^{2}]\tag{33a}$$

and

$$\langle\nu\rangle'_{+}\langle\nu\rangle'_{-}=\tfrac{1}{4}[\langle\nu\rangle'^{\dagger 2}-\langle\nu\rangle'''^{2}].\tag{33b}$$

For $\mu\neq\nu$ we obtain

$$\langle\mu\rangle'^{\dagger}\langle\nu\rangle'''=\langle\mu\rangle'''\langle\nu\rangle'^{\dagger}=\langle\mu\rangle'''\langle\nu\rangle'''=0$$

and hence

$$\langle\mu\rangle'_{\pm}\langle\nu\rangle'_{\pm}=\langle\mu\rangle'_{\pm}\langle\nu\rangle'_{\mp}=\tfrac{1}{4}\langle\mu\rangle'^{\dagger}\langle\nu\rangle'^{\dagger}.\tag{34}$$

We can now state the remaining three algorithms. The first deals with the cases $\langle\mu\rangle_{\pm}\langle\nu\rangle_{\pm}$ and $\langle\mu\rangle_{\pm}\langle\nu\rangle_{\mp}$ for ordinary irreps of A_n.

Algorithm 5.

(1) Evaluate $\langle\mu\rangle\langle\nu\rangle$ for S_n to give

$$\langle\mu\rangle\langle\nu\rangle=g^{\rho}_{\mu\nu}\langle\rho\rangle$$

and restrict the right-hand side to A_n using (15a) and (15b).

(2) If $\langle\mu\rangle \neq \langle\nu\rangle$ then

$$\langle\mu\rangle_\pm\langle\nu\rangle_\pm = \langle\mu\rangle_\pm\langle\nu\rangle_\mp = \tfrac{1}{4}\langle\mu\rangle\langle\nu\rangle.$$

(3) If $\langle\mu\rangle = \langle\nu\rangle$ then evaluate $\langle\mu\rangle''^2$ for S_n:

$$\langle\mu\rangle''^2 = g^{\rho}_{\mu\mu}\langle\rho\rangle$$

where

$$g^{\rho}_{\mu\mu} = \mathrm{i}^{n-k}\chi^{\langle\rho\rangle}_{(p)}.$$

These results are then used in ($31a$) and ($31b$) together with the $\mathrm{S}_n \downarrow \mathrm{A}_n$ branching rules, noting that

$$\langle\mu\rangle\langle\mu\rangle'' = \pm(\langle\mu\rangle_+ - \langle\mu\rangle_-) \qquad \text{as } \chi^{\langle\mu\rangle}_{(p)} = \pm 1.$$

The penultimate algorithm covers the cases $\langle\mu\rangle'_\pm\langle\nu\rangle'_\pm$ and $\langle\mu\rangle'_\pm\langle\nu\rangle'_\mp$.

Algorithm 6.

(1) Evaluate $\langle\mu\rangle'^{\dagger}\langle\nu\rangle'^{\dagger}$ for S_n using algorithm 4 of I to give

$$\langle\mu\rangle'^{\dagger}\langle\nu\rangle'^{\dagger} = g^{\rho}_{\mu\nu}\langle\rho\rangle^{\dagger}$$

and restrict the right-hand side to A_n using ($15a$) and ($15b$).

(2) If $\mu \neq \nu$ then use (34).

(3) If $\mu = \nu$ and μ has unequal parts with one or more even, then evaluate $\langle\mu\rangle''^2$ for S_n, noting that

$$\langle\mu\rangle'''^2 = g^{\rho}_{\mu\mu}\langle\rho\rangle$$

and restricting the right-hand side to A_n using ($15a$) and ($15b$). These results are then used in ($33a$) and ($33b$) to complete the evaluation.

(4) If $\mu = \nu$ and μ has unequal odd parts, evaluate $\langle\mu\rangle'''^2$ for S_n to give

$$\langle\mu\rangle'''^2 = g^{\rho}_{\mu\mu}\langle\rho\rangle$$

where

$$g^{\rho}_{\mu\mu} = \mathrm{i}^{n-k}\chi^{\langle\rho\rangle}_{(\mu)}$$

and restrict the right-hand side to A_n using ($15a$) and ($15b$). Furthermore

$$\langle\mu\rangle'^{\dagger}\langle\mu\rangle''' = \pm(\langle\mu\rangle_+ - \langle\mu\rangle_-) \qquad \text{as } \chi^{\langle\mu\rangle'\dagger}_{(\mu)} = \pm 1.$$

The final algorithm concerns the cases $\langle\mu\rangle_\pm\langle\nu\rangle'_\pm$ and $\langle\mu\rangle_\pm\langle\nu\rangle'_\mp$.

Algorithm 7.

(1) Evaluate $\langle\mu\rangle\langle\nu\rangle'^{\dagger}$ for S_n using algorithm 3 of I to give

$$\langle\mu\rangle\langle\nu\rangle'^{\dagger} = g^{\rho}_{\mu\nu}\langle\rho\rangle'^{\dagger}$$

and restrict the right-hand side to A_n using ($16a$) and ($16b$).

(2) If $\langle\nu\rangle'^{\dagger}$ has unequal parts with one or more even then

$$\langle\mu\rangle_\pm\langle\nu\rangle'_\pm = \langle\mu\rangle_\pm\langle\nu\rangle'_\mp = \tfrac{1}{4}(g^{\rho}_{\mu\nu}\langle\rho\rangle'^{\dagger}).$$

(3) If $\langle\nu\rangle'^{\dagger}$ has unequal odd parts then evaluate

$$\langle\mu\rangle''\langle\nu\rangle''' = g^{\rho}_{\mu\nu}\langle\rho\rangle'^{\dagger}$$

where

$$g^{\rho}_{\mu\nu} = \mathrm{i}^{n-k}\chi^{\langle\rho\rangle'\dagger}_{(\nu)}$$

and restrict to A_n. If

$$\chi^{\langle\nu\rangle'\dagger}_{(\nu)} = \chi^{\langle\mu\rangle\dagger}_{(\nu)}$$

then

$$\langle\mu\rangle^{\dagger}\langle\nu\rangle''' = \langle\mu\rangle''\langle\nu\rangle'^{\dagger} = (-1)^{(n-k)/2}(\langle\nu\rangle'_{+} - \langle\nu\rangle'_{-}),$$

while if

$$\chi^{\langle\nu\rangle'\dagger}_{(\nu)} = -\chi^{\langle\mu\rangle\dagger}_{(\nu)}$$

then $\langle\mu\rangle\langle\nu\rangle''' = -\langle\mu\rangle''\langle\nu\rangle'^{\dagger} = (-1)^{(n-k)/2}(\langle\nu\rangle'_{+} - \langle\nu\rangle'_{-})$. The evaluation is completed by noting (29*a*) and (29*b*).

The above algorithms are essentially free of the need to use character tables. The only ordinary characteristics required are of an especially simple form and may be readily evaluated, if required, using results already given in I (cf Littlewood 1950, p 70) together with equations (20)–(26) of this paper. The spin characteristics $\chi^{\langle\mu\rangle'}_{(p)}$ may be evaluated using the formulae of Morris (1962). No general result seems to be known in the simple sense of Littlewood's theorem.

The application of the above algorithms is best seen in the following examples for A_6. First consider the A_6 product $[51]'_{+}[42]'_{-}$ using algorithm 6. Specialising to S_6, we readily find that

$$[51]'^{\dagger}[42]'^{\dagger} = 2[51]^{\dagger} + 4[42]^{\dagger} + 4[41^2]^{\dagger} + 2[3^2]^{\dagger} + 8[321]^{\dagger}.$$

Use of (13*a*) and (13*b*) for $\mathrm{S}_6\downarrow\mathrm{A}_6$ gives

$$[51]'^{\dagger}[42]'^{\dagger} = 4[51] + 8[42] + 8[41^2] + 4[3^2] + 8[321]_{+} + 8[321]_{-}.$$

Finally, use of (34) results in

$$[51]'_{+}[42]'_{-} = [51] + 2[42] + 2[41^2] + [3^2] + 2[321]_{+} + 2[321]_{-}.$$

Now consider the A_6 product $[51]'_{+}[51]'_{-}$, again using algorithm 6. Using the results of I we find

$$[51]'^{\dagger}[51]'^{\dagger} = [6]^{\dagger} + 2[51]^{\dagger} + 3[42]^{\dagger} + 4[41^2]^{\dagger} + 2[3^2]^{\dagger} + 5[321].$$

Under $\mathrm{S}_6\downarrow\mathrm{A}_6$ the right-hand side becomes

$$2[6] + 4[51] + 6[42] + 8[41^2] + 4[3^2] + 5[321]_{+} + 5[321]_{-}.$$

Furthermore

$$\begin{aligned}[51]'''[51]''' &= [6] - [42] - [321] - [2^2 1^2] + [1^6]\\ &= 2[6] - 2[42] + [321]_{+} + [321]_{-}\end{aligned}$$

and

$$[51]'^{\dagger}[51]''' = -[321]_{+} + [321]_{-}.$$

Use of (33*b*) then yields

$$[51]'_{+}[51]'_{-} = [51] + 2[42] + 2[41^2] + [3^2] + [321]_{+} + [321]_{-}$$

while use of (33*a*) yields

$$[51]'_+[51]'_+ = [6]+[51]+[42]+2[41^2]+[3^2]+[321]_+ + 2[321]_-$$

and

$$[51]'_-[51]'_- = [6]+[51]+[42]+2[41^2]+[3^2]+2[321]_+ + [321]_-.$$

7. Plethysms for ordinary and spin irreps of A_n

We now consider the resolving of the Kronecker squares of the ordinary and spin irreps of A_n into their symmetric and antisymmetric terms.

First we consider the ordinary irreps of A_n that are not members of a conjugate pair. Since in this case $\langle\mu\rangle\downarrow\langle\mu\rangle$ under $S_n\downarrow A_n$, we can simply evaluate $\langle\mu\rangle\otimes\{2\}$ and $\langle\mu\rangle\otimes\{2^2\}$ and $\langle\mu\rangle\otimes\{1^2\}$ as for S_n, using I, and then use equations (15*a*) and (15*b*) to make the $S_n\downarrow A_n$ reductions.

For example, since in S_n we have (cf Butler and King 1973)

$$\langle 2\rangle\otimes\{1^2\} = \langle 1^2\rangle+\langle 1^3\rangle+\langle 21\rangle+\langle 31\rangle,$$

we have for S_7

$$[52]\otimes\{1^2\} = [51^2]+[41^3]+[421]+[3^21]$$

and hence for A_7

$$[52]\otimes\{1^2\} = [51^2]+[41^3]_+ + [41^3]_- + [421]+[3^21].$$

For the conjugate pairs $\langle\mu\rangle_\pm$ of A_n it is necessary to use difference characters and to note that (cf Wybourne 1970)

$$\langle\mu\rangle_\pm\otimes\{2\} = \tfrac{1}{2}[\langle\mu\rangle^\dagger\otimes\{2\}+\langle\mu\rangle''\otimes\{2\}+\langle\mu\rangle^\dagger\langle\mu\rangle''-\langle\mu\rangle_\pm^2] \tag{35a}$$

and

$$\langle\mu\rangle_\pm\otimes\{1^2\} = \tfrac{1}{2}[\langle\mu\rangle^\dagger\otimes\{1^2\}+\langle\mu\rangle''\otimes\{1^2\}+\langle\mu\rangle^\dagger\langle\mu\rangle''-\langle\mu\rangle_\pm^2]. \tag{35b}$$

The ordinary S_n plethysms can be evaluated as in I (cf Butler and King 1973) and the products $\langle\mu\rangle^\dagger\langle\mu\rangle''$ and $\langle\mu\rangle_\pm^2$ as in the previous section. To evaluate $\langle\mu\rangle''\otimes\{2\}$ and $\langle\mu\rangle''\otimes\{1^2\}$, we note that if for A_n

$$\langle\mu\rangle''\langle\mu\rangle'' = 2g^\rho_{\mu\mu}\langle\rho\rangle+\langle\mu\rangle_+ + \langle\mu\rangle_-$$

then

$$\langle\mu\rangle''\otimes\{2\} = g^\rho_{\mu\mu}\langle\rho\rangle+\langle\mu\rangle_\mp \qquad \text{if } \tfrac{1}{2}(n-k) = \begin{matrix}\text{even}\\ \text{odd}\end{matrix} \tag{36a}$$

and

$$\langle\mu\rangle''\otimes\{1^2\} = g^\rho_{\mu\mu}\langle\rho\rangle+\langle\mu\rangle_\pm \qquad \text{if } \tfrac{1}{2}(n-k) = \begin{matrix}\text{even}\\ \text{odd.}\end{matrix} \tag{36b}$$

Use of the above results readily leads to

$$[321]_\pm\otimes\{1^2\} = 2[41^2]+[321]_\pm$$

for A_6.

For the spin irreps of A_n we need to treat the two cases $n=2\nu$ and $n=2\nu+1$ separately. The primary need is to evaluate the plethysms for the basic spin irrep $\langle 0\rangle'^{\dagger}$, since any other plethysm involving spin irreps can be reduced to a plethysm involving the basic spin irreps and those involving ordinary irreps.

In the case of $A_{2\nu}$ the squares of the basic spin irreps are resolved by use of (86*a*) and (86*b*) of I followed by use of the $S_n \downarrow A_n$ branching rules. If $\nu = 0, 1 \pmod 4$ the basic spin irrep of $A_{2\nu}$ is orthogonal, while if $\nu = 2, 3 \pmod 4$ it is symplectic.

In the case of $A_{2\nu+1}$ difference characters for the basic spin irrep are used exactly as in (35*a*)–(35*b*), leading to the conclusion that if $\nu = 1, 3 \pmod 4$, $\langle 0\rangle'_{\pm}$ are complex while if $\nu = 0 \pmod 4$, $\langle 0\rangle_{\pm}$ are orthogonal and if $\nu = 2 \pmod 4$, $\langle 0\rangle'_{\pm}$ are symplectic.

8. Classification of the A_n irreps

The classification of the irreps of A_n as to their complex, orthogonal or symplectic characters follows immediately from the plethysm results just outlined. We simply quote the final results in the form of two algorithms.

The first algorithm is for ordinary irreps of A_n.

Algorithm 8.

(1) If $[\lambda] = [\tilde{\lambda}]$ in S_n and $(n-k) \neq 0 \pmod 4$ then $[\lambda]_{\pm}$ of A_n is complex.

(2) All other ordinary irreps of A_n are real and orthogonal.

In using the above algorithm k is the number of $p_i = 2\lambda_i - 2i + 1$ with $p_i > 0$.

The second algorithm is for the spin irreps of A_n.

Algorithm 9.

(1) If $(n-k)/2$ is odd then the spin irreps $[\lambda_1\lambda_2 \ldots \lambda_k]'_{\pm}$ are complex.

(2) If $n-k+1$ or $(n-k)/2$ are even then for $n = 2\nu+1$ we have for the spin irreps

$\nu = 0, 3 \pmod 4$ orthogonal,

$\nu = 1, 2 \pmod 4$ symplectic,

while for $n = 2\nu$ we have

$\nu = 0, 1 \pmod 4$ orthogonal,

$\nu = 2, 3 \pmod 4$ symplectic.

9. Concluding remarks

The algorithms developed herein allow any Kronecker product in A_n to be unambiguously resolved, essentially without the need for extensive character tables. The reduced notation permits the results to be displayed in an n-independent manner and there should be little difficulty in implementing the algorithms as a computer program.

Acknowledgment

We are appreciative of the work done by Janet Warburton in preparing this manuscript for publication.

References

Boerner H 1970 *Representations of Groups* 2nd edn (Amsterdam: North-Holland) p 207
Butler P H and King R C 1973 *J. Math. Phys.* **14** 1176
Curtis C W and Reiner I 1962 *Representation Theory of Finite Groups and Associations Algebras* (New York: Interscience)
Davies J W and Morris A O 1974 *J. London Math. Soc. Ser. 2* **8** 615
Dornhoff L 1971 *Group Representation Theory* Part A (New York: Marcel Dekker)
Frobenius G 1901 *S.B. Preuss Akad. Wiss.* 303
Hamermesh M 1962 *Group Theory and its Applications to Physical Problems* (London: Addison-Wesley) p 25
Hardy G H and Wright E M 1954 *An Introduction to the Theory of Numbers* 3rd edn (Oxford: University Press) p 279
Lax M 1974 *Symmetry Principles in Solid State and Molecular Physics* (New York: Wiley) p 50
Ledermann W 1976 *Introduction to Group Theory* (London: Longmans) p 139
Littlewood D E 1950 *The Theory of Group Characters* (Oxford: Clarendon)
Luan Dehuai and Wybourne B G 1981 *J. Phys. A: Math. Gen.* **14** 327
Morris A O 1962 *Proc. London Math. Soc. Ser. 3* **12** 55
Puttaswamaiah B M and Robinson G de B 1964 *Can. J. Math.* **16** 587
Read E W 1977 *J. Algebra* **46** 102
Schur I 1911 *J. Reine angew. Math.* **139** 155
Sharp W T, Biedenharn L C, de Vries E and van Zanten A J 1975 *Can. J. Math.* **27** 246
Wybourne B G 1970 *Symmetry Principles and Atomic Spectroscopy* (New York: Wiley) p 134